6.01 Introduction to EECS via Robotics

Lecture 4: Analyzing System Behavior

Lecturer: Adam Hartz (hz@mit.edu)

As you come in...

- Grab one handout (on the table by the entrance)
- Please sit near the front!

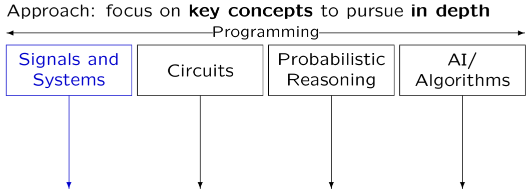
Notes

6.01: Big Ideas

The **intellectual themes** in 6.01 are recurring themes in engineering:

- design of complex systems
- modeling and controlling physical systems
- augmenting physical systems with computation
- building systems that are robust to uncertainty

Approach: focus on **key concepts** to pursue **in depth**



Focus: discrete-time feedback control systems

Notes

This week

Today: Last signals and systems lecture :(
Labs: Fixing wall follower

Notes

Midterm 1

Time: Tuesday, 12 March, 7:30-9:30pm

Room: TBD

Coverage: Everything up to and including week 5

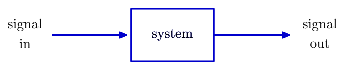
You may refer to any printed materials you bring.
You may not use computers, phones, or calculators.

Review materials will be posted this weekend.

Notes

The Signals and Systems Abstraction

Describe a **system** (physical, mathematical, or computational) by the way it transforms an input signal into an output signal.



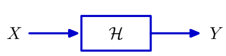
Multiple representations with different strengths:

- **Difference Equation**: concise mathematical representation
- **State Machine**: computational framework for simulation
- **Block Diagram**: visual representation of signal flow paths
- **Operator Equation**: manipulation and combination
- **System Functional**: represent systems as operators
- **Poles**: predict long-term behavior

Notes

System Functional – Review

We can express the relation between the (known) input and the (unknown) output using the system functional $\mathcal{H}$.



The system functional $\mathcal{H}$ is an operator.

Applying $\mathcal{H}$ to X yields Y.

$$Y = \mathcal{H}X$$

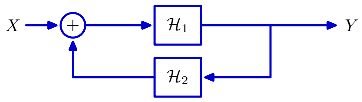
It is also convenient to think of $\mathcal{H}$ as a ratio:

$$\mathcal{H} = \frac{Y}{X}$$

Notes

Feedback and Cyclic Flow Paths

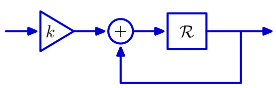
Feedback produces cyclic flow paths, which lead to persistent responses for transient inputs.

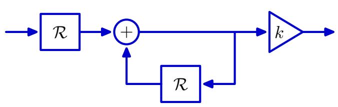


Notes

Check Yourself!

Are the following systems *equivalent*?

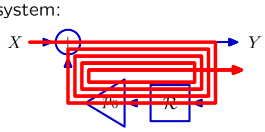




Notes

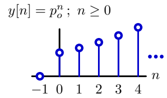
Feedback and Cyclic Signal Flow Paths

Consider the following system:



$$\frac{Y}{X} = \frac{1}{1 - p_0\mathcal{R}} = 1 + p_0\mathcal{R} + p_0^2\mathcal{R}^2 + p_0^3\mathcal{R}^3 + p_0^4\mathcal{R}^4 + \dots$$

Characterize persistent responses (*modes*) to transient (e.g. unit sample) inputs with **poles**.



Notes

Finding Poles – Review

Poles can be identified by factoring the denominator of the system functional:

$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + \ldots}{1 + a_1\mathcal{R} + a_2\mathcal{R}^2 + \ldots}$$

$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + \ldots}{(1 - p_0\mathcal{R})(1 - p_1\mathcal{R})(1 - p_2\mathcal{R})\ldots}$$

The p_i values are the *poles*, and one geometric mode p_i^n arises from each pole.

Notes

Finding Poles – Review

$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + \ldots}{(1 - p_0\mathcal{R})(1 - p_1\mathcal{R})(1 - p_2\mathcal{R})\ldots}$$

Partial fraction expansion:

$$\frac{Y}{X} = \frac{c_0}{1 - p_0\mathcal{R}} + \frac{c_1}{1 - p_1\mathcal{R}} + \frac{c_2}{1 - p_2\mathcal{R}} + \ldots + f_0 + f_1\mathcal{R} + f_2\mathcal{R}^2 + \ldots$$

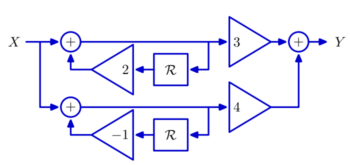
If the system functional is a *proper* rational polynomial, then the unit sample response is:

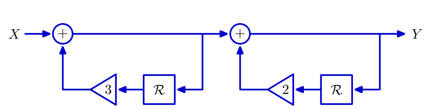
$$y[n] = c_0 p_0^n + c_1 p_1^n + c_2 p_2^n + \ldots$$

Notes

Check Yourself!

What are the poles of the following systems?





Notes

Long-term Behavior: Dominant Pole – Review

When analyzing systems' poles, we are interested in **long-term** behavior (not specific samples).

As $n \to \infty$, how does $y[n]$ behave?

We have seen that a system's unit sample response can be written in the form:

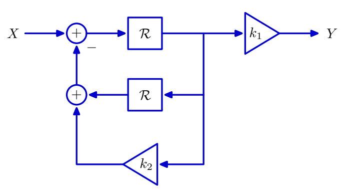
$$y[n] = \sum_k c_k p_k^n$$

In the "large-n" case, all poles but the one with the largest magnitude die away, and so looking at the dominant pole alone tells us about the behavior of the system in that case.

Notes

Check Yourself!

Consider the following system:



Answer the following questions:

1. How many poles does this system have?
2. Will changing k_1 affect the system's poles?
3. Will changing k_2 affect the system's poles?
4. Will changing k_1 affect the system's unit sample response?
5. Will changing k_2 affect the system's unit sample response?

Notes

Complex Poles – Review

What if a pole has a non-zero imaginary part?

Example:

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} + \mathcal{R}^2}$$

Poles at $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}j$.

Unit sample response still goes like poles raised to the power n!

Need to understand what happens when complex numbers are raised to integer powers.

Notes

Complex Poles – Review

Easiest to understand when poles are represented in *polar form*:

A number $p_0 = a_0 + b_0 j$ can be represented by a magnitude and an angle in the complex plane:

$$a_0 + b_0 j = r(\cos(\theta) + j\sin(\theta))$$

where $r = \sqrt{a_0^2 + b_0^2}$ and $\theta = \tan^{-1}(b_0, a_0)$

By Euler's formula:

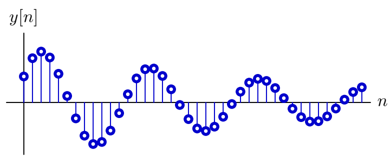
$$a_0 + b_0 j = re^{j\theta}$$

Furthermore, we can express $(re^{j\theta})^n$ as $r^n e^{jn\theta}$. This is a complex number with magnitude r^n and angle $n\theta$. (Thus each new power multiplies previous magnitude by r and adds to previous angle by θ.)

Notes

Check Yourself!

Output of a system with poles at $z = re^{\pm j\omega}$



Which statement is true?

1. $r < 0.5$ and $\omega \approx 0.5$
2. $0.5 < r < 1$ and $\omega \approx 0.5$
3. $r < 0.5$ and $\omega \approx 0.08$
4. $0.5 < r < 1$ and $\omega \approx 0.08$
5. None of the above

Notes

Summary: Pole Behaviors – Review

Unit sample response of most systems can be reduced to the form:
$y[n] = \sum_i c_i p_i^n$

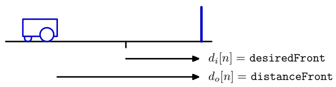
In the long term, the response of the pole with the largest magnitude dominates the overall response.

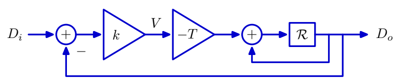
Can figure out properties of the response by thinking about geometric sequences!

Notes

Example: Wall Finder

Consider a variant of the wall finder from week 2:

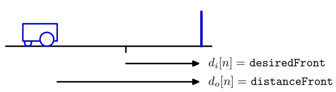


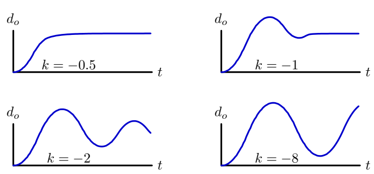


Notes

Example: Wall Finder

Using feedback to control position (DL02) can lead to bad behaviors:

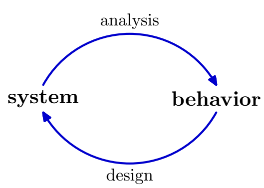




What causes these different types of responses?
Is there a systematic way to optimize the gain k?

Notes

Design

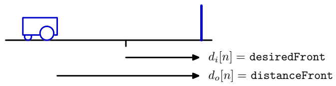


The most useful tools are those that help us not only analyze systems, but also *design* systems

Notes

Example: Wall Finder

The difference equations provide a concise description of behavior:



proportional controller: $v[n] = ke[n] = k\big(d_i[n] - d_s[n]\big)$
locomotion: $d_o[n] = d_o[n-1] - Tv[n-1]$
sensor with no delay: $d_s[n] = d_o[n]$

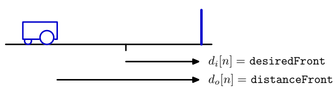
$$d_o[n] = d_o[n-1] - Tk(d_i[n-1] - d_o[n-1])$$

However, it provides little insight into how to choose k.

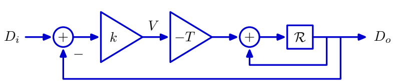
Notes

Example: Wall Finder

A block diagram reveals two feedback paths:



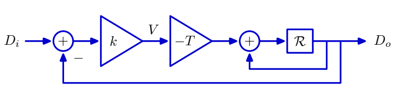
proportional controller: $v[n] = ke[n] = k\big(d_i[n] - d_s[n]\big)$
locomotion: $d_o[n] = d_o[n-1] - Tv[n-1]$
sensor with no delay: $d_s[n] = d_o[n]$



However, it provides little insight into how to choose k.

Notes

Check Yourself!



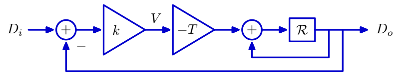
To construct this system using our simulation framework, how many instances of `Cascade` and `FeedbackAdd`/`FeedbackSubtract` are needed?

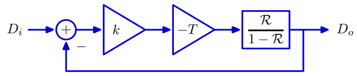
1. 1 `Cascade`, 1 `Feedback`
2. 2 `Cascade`, 1 `Feedback`
3. 1 `Cascade`, 2 `Feedback`
4. 2 `Cascade`, 2 `Feedback`
5. None of the above

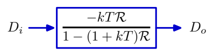
Notes

Example: Wall Finder

Simplify block diagram with $\mathcal{R}$ operator and system functionals.







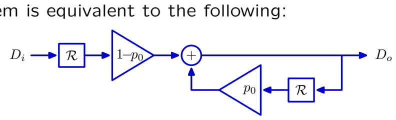
Notes

Example: Wall Finder

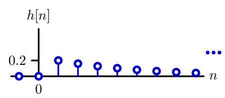
This system contains a single pole at $z = 1 + kT$.

$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1-(1+kT)\mathcal{R}}$$

The whole system is equivalent to the following:



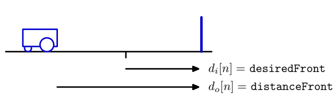
where $p_0 = 1 + kT$.
Unit sample response for $kT = -0.2$ is:



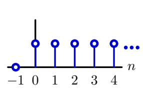
Notes

Example: Wall Finder

We are often interested in the *step response* of a control system.



Idea: start the output $d_o[n]$ at zero while the input is held constant at 1.

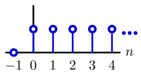


Notes

Step Response

We can think of the unit-step signal $u[n]$ as an accumulation of a series of samples $\delta[n]$

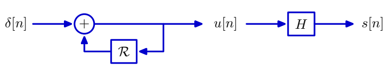
$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \dots$



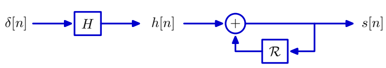
Notes

Step Response

Unit step response $s[n]$ is response of $\mathcal{H}$ to the unit-step signal $u[n]$, which is constructed by accumulation of the unit sample signal $\delta[n]$.



Commute and relabel signals:



The unit-step response $s[n]$ is equal to the accumulated unit sample response $h[n]$:

$$s[n] = \sum_{i=-\infty}^{n} h[i]$$

Notes

Example: Wall Finder

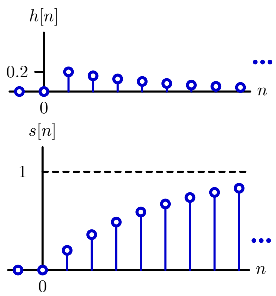
We can use this idea to see how our wall finder converges to a target distance:

- decide on a target distance d
- then input to system would just be $d \cdot u[n]$
- so just analyzing response of system to $u[n]$ will provide insight into speed of convergence and behavior of convergence
- and we just saw that is simply the sum of the unit sample responses

Notes

Example: Wall Finder

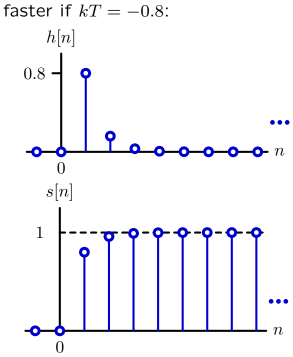
The step response of the `wallFinder` system with $kT = -0.2$ is slow because the unit-sample response is slow (remember in this case that pole is at $p_0 = 1 + kT$, initial response is $1 - p_0 = -kT$, decay is p_0):



Notes

Example: Wall Finder

The step response is faster if $kT = -0.8$:

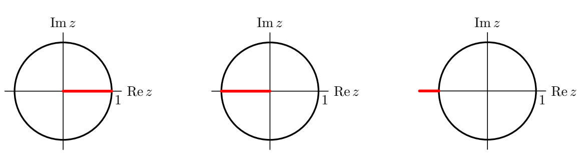


Notes

Wall Finder: Poles

The poles of the system provide insight for choosing k!

$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1-(1+kT)\mathcal{R}} = \frac{(1-p_0)\mathcal{R}}{1-p_0\mathcal{R}}; \qquad p_0 = 1 + kT$$



$-1 < kT < 0$	$-2 < kT < -1$	$kT < -2$
$0 < p_0 < 1$	$-1 < p_0 < 0$	$p_0 < -1$
monotonic converging	alternating converging	alternating diverging

Notes

Check Yourself!

$$\frac{D_o}{D_i} = \frac{-kT\mathcal{R}}{1-(1+kT)\mathcal{R}}$$

Which value of kT gives the fastest convergence of the unit-sample response?

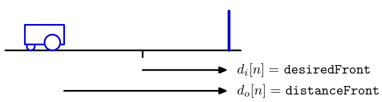
1. $kT = -2$
2. $kT = -1$
3. $kT = 0$
4. $kT = 1$
5. $kT = 2$

0. None of the above

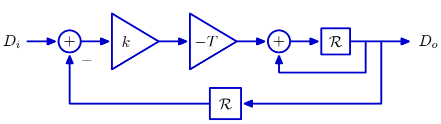
Notes

Example: Wall Finder *with Delay*

Incorporating sensor delay in block diagram:

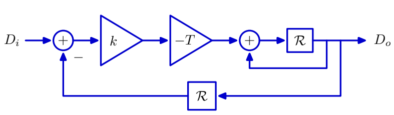


proportional controller:	$v[n] = ke[n] = k\big(d_i[n] - d_s[n]\big)$
locomotion:	$d_o[n] = d_o[n-1] - Tv[n-1]$
sensor with delay:	$d_s[n] = d_o[n-1]$



Notes

Example: Wall Finder *with Delay*



What is the system functional $\frac{D_o}{D_i}$?

1. $\dfrac{kT\mathcal{R}}{1-\mathcal{R}}$
2. $\dfrac{-kT\mathcal{R}}{1+\mathcal{R}+kT\mathcal{R}^2}$
3. $\dfrac{-kT\mathcal{R}}{1-\mathcal{R}-kT\mathcal{R}^2}$
4. $\dfrac{kT\mathcal{R}}{1-\mathcal{R}} + kT$

Notes

Example: Wall Finder *with Delay*

Substitute $\frac{1}{z}$ for $\mathcal{R}$ to find the poles.

$$\frac{Y}{X} = \frac{-kT\mathcal{R}}{1-\mathcal{R}-kT\mathcal{R}^2}$$

$$\frac{Y}{X} = \frac{-kT\frac{1}{z}}{1-\frac{1}{z}-kT\left(\frac{1}{z}\right)^2}$$

$$\frac{Y}{X} = \frac{-kTz}{z^2-z-kT}$$

The poles are the roots of the denominator in z:

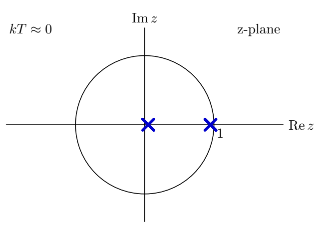
$$z = \frac{1}{2} \pm \sqrt{\frac{1}{4}+kT}$$

Notes

Example: Wall Finder *with Delay*

For small kT, the poles are at $z \approx -kT$ and $z \approx 1+kT$.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT} = \frac{1}{2}\left(1 \pm \sqrt{1+4kT}\right) \approx \frac{1}{2}(1 \pm (1+2kT)) = 1+kT,\ -kT$$



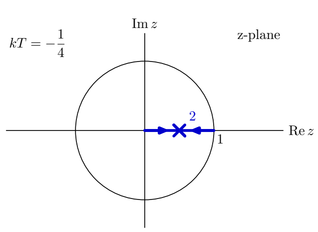
Pole near 0 generates fast response. Pole near 1 generates slow response.
Slow mode dominates the response.

Notes

Example: Wall Finder *with Delay*

As kT becomes more negative, the poles move toward each other and collide at $z = \frac{1}{2}$ when $kT = -\frac{1}{4}$.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT} = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 - \frac{1}{4}} = \frac{1}{2}, \frac{1}{2}$$

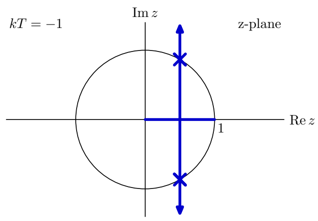


Notes

Example: Wall Finder *with Delay*

If $kT < -\frac{1}{4}$, the poles are complex.

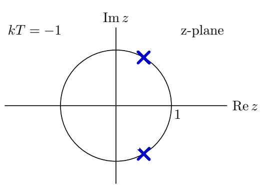
$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT} = \frac{1}{2} \pm j\sqrt{-kT - \left(\frac{1}{2}\right)^2}$$



Complex poles → oscillations.

Notes

Check Yourself!



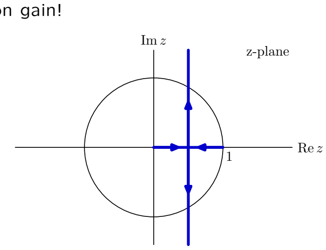
What is the period of the oscillation?

1. 1
2. 2
3. 3
4. 4
5. 6

Notes

Feedback and Control: Poles

The poles depend on gain!

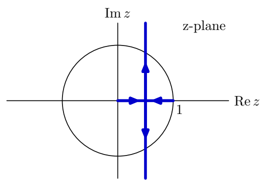


If $kT : 0 \to -\infty$: then $(z_1, z_2) : (0, 1) \to (\frac{1}{2}, \frac{1}{2}) \to (\frac{1}{2} \pm j\infty)$

Our design problem can be thought of as choosing k to move the poles to a "desirable" location in the complex plane.

Notes

Check Yourself!



closed-loop poles

$$\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + kT}$$

Find kT for fastest response.

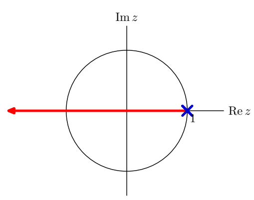
1. 0
2. $-1/4$
3. $-1/2$
4. -1
5. $-\infty$
0. None of the above

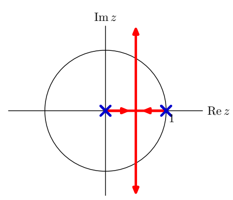
Notes

Effect of Delay

Adding delay to the feedback loop makes the system more difficult to stabilize.

Ideal sensor: $d_s[n] = d_o[n]$
More realistic sensor: $d_s[n] = d_o[n-1]$





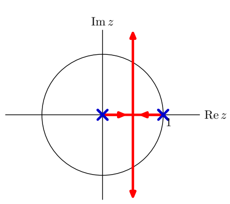
Fastest response without delay: single pole at 0
Fastest with delay: double pole at 0.5 (**slower!**)

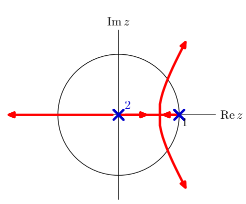
Notes

Effect of Delay

Adding more delay in the feedback loop is even worse.

More realistic sensor: $d_s[n] = d_o[n-1]$
Doubly-delayed sensor: $d_s[n] = d_o[n-2]$

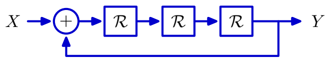




Fastest with delay: double pole at 0.5
Fastest with two delays: double pole at 0.682 (**slower!**)

Notes

Check Yourself!



Which of the following statements are true?

1. The system has 3 poles.
2. Unit-sample Response is the sum of 3 geometric sequences.
3. Unit-sample Response is $y[n] = [0, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots]$
4. Unit-sample Response is $y[n] = [1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots]$
5. One of the poles is at $z = 1$.

Notes

Summary

System functionals provide a convenient summary of information that is important for designing control systems.

The unit sample response of a feedback system is the sum of scaled geometric sequences whose bases are the system's poles.

The long-term behavior of a system is determined by its dominant pole (i.e, the pole with the largest magnitude).

This Week's Labs: Fixing Wall Follower

Notes

Notes