Lecture 5: Circuits Introduction

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As you come in...

- Grab one handout (on the table by the entrance)
- If you plan on using a laptop or smartphone during lecture, please sit near the back.
Focus on **key concepts**

- AI/algorithms
- feedback and control
- circuits
- probability and planning

Modular Design of Complex Systems
Focus on **key concepts** with **explicit connections**

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Focus on **key concepts** with **explicit connections**

AI/algorithms  \[\rightarrow\]  feedback and control  \[\leftarrow\]  probability and planning  \[\downarrow\]  circuits

in an **authentic context** with overarching theme: **Modular Design of Complex Systems**
Module 1: Signals and Systems

Focus on:
- **modeling** and **simulation** of physical systems
- augmenting physical systems with **computation**

Topics: Discrete-time LTI Feedback Control Systems

Lab Exercises: Robotic Driving, “Jousting”
Module 1: Signals and Systems

Controlling complexity through modularity and abstraction:

Python:

- Primitives: +, *, ==, !=, ...
- Combination: if, while, f(g(x)), ...
- Abstraction: def, class, ...

LTI:

- Primitives: Gains, Delays
- Combination: Adders, Cascade, Feedback, ...
- Abstraction: System Functionals
Module 2: Circuits

Focus on:
- Designing, constructing, and analyzing physical systems

Topics: Resistive Networks, Op-Amps, Equivalence

Lab Exercises: Design new sensory modality for the robot
**The Circuit Abstraction**

Circuits represent systems as *components* connected by *nodes*.

Currents flow through components, and Voltages develop across components.
Quantities of Interest

- **Voltage**: The difference in electrical potential energy between two points. Measured in Volts (V, J/C)
- **Current**: The rate of flow of positive charge past a point. Measured in Amperes (Amps, A, C/s)
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It is the *difference* in potential between two nodes that drives the currents.
The Circuit Abstraction

Circuits are useful and important for (at least) two very different reasons:

- as **models** of complex systems
  - biological models
  - thermodynamic models
  - fluid models

- as **physical systems**
  - power (generators, transformers, power lines, etc)
  - electronics (cell phones, computers, etc)
  - sensors (sonars, glucose sensors, etc)
Example: Flashlight

We can represent a flashlight as a voltage source (battery) connected to a resistor (light bulb).

The voltage source generates a voltage $v$ across the resistor and a current $i$ through the resistor.
Example: Myelinated Fiber

Model of myelinated nerve fiber
The **primitives** are simple elements: sources and resistors. The **rules of combination** are the rules that govern the flow of current and the development of voltage.

\[ v = V_0 \quad i = -I_0 \quad v = iR \]
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\[ v = V_0 \]

\[ i = -I_0 \]

\[ v = iR \]
Circuits: Primitives

Resistor

Circuits: Primitives

Voltage Source


\[ i \uparrow \quad + \quad \quad + \]

\[ V_0 \quad - \quad v \quad - \]

\[ v = V_0 \]
Circuits: Primitives

Current Source

\[ i = -I_0 \]

Example 1:

The voltage source determines the voltage across the resistor, so the current through the resistor is $i = \frac{v}{R} = \frac{1V}{1\Omega} = 1A$. 

![Circuit Diagram](image)
Analyzing More Complex Circuits

More complicated circuits are more complicated to analyze, but can be analyzed systematically by applying constitutive equations and two conservation laws.
Analyzing Circuits: KVL

KVL: The sum of the voltages around any closed path is zero.

\[
\begin{align*}
  v_1 &= V_0 \\
  v_2 + v_3 + v_4 + v_5 + v_6 &= 0
\end{align*}
\]
Analyzing Circuits: KVL

KVL: The sum of the voltages around any closed path is zero.

Example: $-v_1 + v_2 + v_4 = 0$
Impossible Things

Penrose Staircase

\[3V - 6V - 2V \neq 0V\]
The flow of electrical current is analogous to the flow of incompressible fluid (e.g., water):

All water that flows into a junction must flow out. All current that flows into a node must flow out.

Current $i_1$ flows in, and two currents $i_2$ and $i_3$ flow out:

$$i_1 = i_2 + i_3$$
Analyzing Circuits: KCL

KCL: The net flow of current into (or out of) a node is zero.

\[ \sum i_{in} = \sum i_{out} \]
Analyzing Circuits: KCL

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Here, there are two nodes. The net current out of each must be zero.
Analyzing Circuits: KCL

**KCL:** The net flow of current into (or out of) a node is zero.

\[ \sum i_{in} = \sum i_{out} \]

Here, there are two nodes. The net current out of each must be zero.

For the top node: \( i_1 + i_2 + i_3 = 0 \).
KCL: The net flow of current into (or out of) a node is zero.

\[ \sum i_{in} = \sum i_{out} \]

Here, there are two nodes. The net current out of each must be zero.

For the top node: \( i_1 + i_2 + i_3 = 0 \).
Same for the bottom node.
At each node, KCL must hold.

We can write this relationship for each node:
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\[ i_1 + i_2 = 0 \]
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\[ i_2 = i_3 + i_4 \]
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\[ i_2 = i_3 + i_4 \]
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Analyzing More Complex Circuits

More complicated circuits are more complicated to analyze, but they can be analyzed systematically by applying constitutive equations and two conservation laws.

![Circuit Diagram]

One strategy:
- Give every current and every node potential a name
- Choose a node potential to be our reference 0V
- Write one equation per component (7)
- Write one KCL equation per node, except for the reference node (4)

12 equations in 12 unknowns!
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12 equations in 12 unknowns! :(
Analyzing More Complex Circuits

We will still need to solve all of these equations (to figure out the voltages and currents throughout the circuit), but there may be an easier way.

• pick a node to be reference (0V)

- Find a component equation or KCL equation with exactly one unknown, and solve for that value directly (it now becomes a known value).
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(Version 0.0.1)
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(Version 0.0.1)

- pick a node to be reference (0V)
- repeat until circuit solved:
  - Find a component equation or KCL equation with exactly one unknown, and solve for that value directly (it now becomes a known value).
Example: Complicated Circuit

\[ V_1 + \quad 4\Omega \quad 5\Omega \quad 1\Omega \quad 2\Omega \quad V_2 \]

\[ I_1 \quad 3A \quad I_2 \]

\[ I_1 = 3A \quad I_2 = 3A \]
What is the current $i$ through the resistor?

1. 1A
2. 2A
3. 0A
4. cannot be determined
5. none of the above
Check Yourself!

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Find the voltage $V_o$ in the circuit below:

0. 14V
1. 10V
2. 9V
3. 6V
4. something else
5. can’t be solved (contradiction)
Check Yourself!

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Example: Complicated Circuit
Abstractions: “One-Port”

A “one-port” is a circuit that can be represented as a single element:

Current enters one terminal (+) and leaves the other (-). The one-port constrains the relationship between the current $i$ and the voltage $v$. 
Abstractions: Series Combination

Components can be combined in “series” to form new one-ports:

\[ V = v_1 + v_2 \]
\[ I = i_1 = i_2 \]
Components can be combined in “parallel” to form new one-ports:

\[ V = v_1 = v_2 \]
\[ I = i_1 + i_2 \]
Check Yourself!

If the two boxed circuits have the same $v/i$ relationship, which of the following, is true?

1. $R_s < R_1$ and $R_s < R_2$
2. $R_1 < R_s < R_2$
3. $R_2 < R_s < R_1$
4. $R_s > R_1$ and $R_s > R_2$
5. None of the above
Series Resistors

The series combination of two resistors is equivalent to a single resistor whose resistance is the sum of the two original resistances.

\[ v = iR_2 + iR_1 = iR_s \]
\[ R_s = R_1 + R_2 \]

The series equivalent resistance is always **larger** than either of the original resistances.
If the two boxed circuits have the same $v/i$ relationship, which of the following, is true?

1. $R_p < R_1$ and $R_p < R_2$
2. $R_1 < R_p < R_2$
3. $R_2 < R_p < R_1$
4. $R_p > R_1$ and $R_p > R_2$
5. None of the above
Parallel Resistors

The parallel combination of two resistors is equivalent to a single resistor whose conductance (1/resistance) is the sum of the two original conductances.

\[ i = \frac{v}{R_1} + \frac{v}{R_2} = \frac{v}{R_p} \]

\[ \frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} \]

\[ R_p = R_1 || R_2 = \frac{R_2 R_1}{R_2 + R_1} = \frac{R_1}{1 + \frac{R_2}{R_1}} = \frac{R_2}{1 + \frac{R_1}{R_2}} \]

The parallel equivalent resistance is always smaller than either of the original resistances.
Check Yourself!

What is the equivalent resistance of this one-port?

1. 0.5Ω
2. 1Ω
3. 2Ω
4. 3Ω
5. 5Ω
Check Yourself!

What is the equivalent resistance of this one-port?

1. 0.5Ω
2. 1Ω
3. 2Ω
4. 3Ω
5. 5Ω
Check Yourself!

One curve represents the equivalent resistance of $R$ in parallel with $10 \Omega$, and the other represents the equivalent resistance of $R$ in series with $10 \Omega$. Which is which?
1. Pick a node to be our reference node. All other node potentials will be measured with respect to this node.

2. Look for a constitutive equation with exactly one unknown value. If such an equation exists, solve for the unknown value and GOTO 5.

3. Look for a KCL equation with exactly one unknown current. If such an equation exists, solve for the unknown current and GOTO 5.

4. If no equation with exactly one unknown, look for patterns that can simplify the circuit (series/parallel combinations, etc), and GOTO 2.

5. If the circuit is completely solved, congratulations! If not, GOTO 2.
Example: Complicated Circuit

\[
10V
\]

\[
1k\Omega
\]

\[
1k\Omega
\]

\[
1k\Omega
\]
Resistors in series act as voltage dividers:

\[ I = \frac{V}{R_1 + R_2} \]

\[ V_1 = R_1 I = \frac{R_1}{R_1 + R_2} V \]

\[ V_2 = R_2 I = \frac{R_2}{R_1 + R_2} V \]
Resistors in parallel act as current dividers:

\[ V = (R_1 || R_2)I \]

\[ I_1 = \frac{V}{R_1} = \frac{1}{R_1} \frac{R_1 R_2}{R_1 + R_2} I = \frac{R_2}{R_1 + R_2} I \]

\[ I_2 = \frac{V}{R_2} = \frac{1}{R_2} \frac{R_1 R_2}{R_1 + R_2} I = \frac{R_1}{R_1 + R_2} I \]
Check Yourself!

Which of the following is true?

1. $V_o \leq 3V$
2. $3V < V_o \leq 6V$
3. $6V < V_o \leq 9V$
4. $9V < V_o \leq 12V$
5. $V_o > 12V$
Which of the following is true?

1. $V_o \leq 3V$
2. $3V < V_o \leq 6V$
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4. $9V < V_o \leq 12V$
5. $V_o > 12V$
Combining earlier ideas, we can develop a process by which we can solve for all currents and potentials in a circuit.

1. Pick a node to be our reference node. All other node potentials will be measured with respect to this node.

2. Look for a constitutive equation with exactly one unknown value. If such an equation exists, solve for the unknown value and GOTO 6.

3. Look for a KCL equation with exactly one unknown current. If such an equation exists, solve for the unknown current and GOTO 6.

4. If no equation with exactly one unknown, look for patterns that can simplify the circuit (series/parallel combinations, etc), and GOTO 2.

5. **Last Resort:** If no simplifications, write a small system of constitutive and KCL equations in terms of node potentials, and solve. GOTO 6.

6. If the circuit is completely solved, congratulations! If not, GOTO 2.
Check Yourself

Solving for $V_1$ in the circuit below, Ben Bitdiddle takes the following steps.

1. Because the $20\,V$ source is connected to the $4\,\Omega$ resistor, Ben knows that the current flowing through that resistor must be $20\,V/4\,\Omega = 5\,A$.
2. By KCL, that same current must be flowing through the $1\,\Omega$ resistor.
3. $5\,A$ through the $1\,\Omega$ resistor means that the voltage drop across the resistor must be $5\,A \cdot 1\,\Omega = 5\,V$.
4. The voltage drop across the $1\,\Omega$ resistor is exactly the $V_1$ Ben wanted to compute, so $V_1 = 5\,V$.

Where was Ben’s first mistake, if any?
Check Yourself

Solving for $V_2$ in the circuit below, Ben Bitdiddle takes the following steps.

1. Ben notes that the two resistors are connected in parallel and decides to replace them with a single equivalent resistor whose resistance is $4\Omega || 12\Omega = 3\Omega$.
2. The current through the combination is $9\text{V}/3\Omega = 3\text{A}$.
3. Re-expanding that combination, the current flowing through each resistor must be $3\text{A}$.
4. $V_2$ is the drop across the $2\Omega$ resistor. Because $3\text{A}$ is flowing through it, $V_2 = 2\Omega \cdot 3\text{A} = 6\text{V}$.

Where was Ben's first mistake, if any?
Check Yourself

Solving for $V_3$ in the circuit below, Ben Bitdiddle takes the following steps.

1. Ben notes that there is a $2V$ drop across the total resistance on the top, which consists of the $2\Omega$ resistor in series with the parallel combination of $6\Omega$ and $10\Omega$.
2. The equivalent resistance of the parallel combination of $6\Omega$ and $10\Omega$ is $6\Omega \| 10\Omega = 8\Omega$.
3. The total resistance is, therefore, $2\Omega + 8\Omega = 10\Omega$.
4. By Ohm’s Law, the total current is $2V/10\Omega = 0.2A$, flowing left-to-right.
5. The voltage drop across the parallel combination, then, is $0.2A \cdot 8\Omega = 1.6V$.
6. The total voltage drop $V_4$ is $8V + 1.6V = 9.6V$.

Where was Ben’s first mistake, if any?
Interaction of Circuit Elements

Circuit design is complicated by interactions among the elements. Adding an element changes voltages and current *throughout* the circuit.

Example: closing a switch is equivalent to adding a new element.
Check Yourself!

How does closing the switch affect $V_o$ and $I_o$?

1. $V_o$ decreases, $I_o$ decreases
2. $V_o$ decreases, $I_o$ increases
3. $V_o$ increases, $I_o$ decreases
4. $V_o$ increases, $I_o$ increases
5. depends on bulb’s resistance
Check Yourself!

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4. $V_o$ increases, $I_o$ increases
5. depends on bulb's resistance
Summary

This time:

• Defined the circuit abstraction (“through” and “across” variables)
• Developed systematic way of solving circuits
• Developed means of thinking about circuits through patterns (series/parallel) and abstractions (one-port)
• Noticed that the ways in which we think about abstraction and modularity in circuits is fundamentally different from the way we thought about these ideas in LTI and programming

Labs This Week:

• *Software Hardware Lab*: Dividers, Breadboarding
• *Design Lab*: Joystick-controlled robot!!!