The World is in Continuous Time

\[ y(t) = A_1 e^{-t/\tau} + A_2 e^{t/\tau} + \ldots \]

- The world rotates continuously
- Temperatures vary continuously
- We’re trained to think in terms of seconds
- Why do signals and systems analysis in discrete time? \[ y[n] \] and not \[ y(t) \] ???

Why Discrete Time?

- Computers/Digital Systems work in steps/iterations
- Discrete time is how you think about steps

Clock

Activity

Signals
**Discrete in a Continuous World**

- Digital Systems must analyze, model, and act in discrete time, but they must work with continuous time systems (humans, physics, etc).
- As long as sample period $T$ is much shorter than any "features" you're usually good...otherwise care needs to be taken.

**Signals:**
- Time Steps $(n)$
- Time $(s)$

**LTI: Linear Time Invariant Systems**

- **Linear:** If a system $H$ responds to a signal $X$ with output $Y$ then system $H$ will respond to a signal $2X$ with output $2Y$.

**Start:**
- $X$
- $Y$

**Scaled:**
- $2X$
- $2Y$

**LTI: Linear Time Invariant Systems**

- **Time-Invariant:** If a system $H$ responds to a signal $X$ with output $Y$ then system $H$ will respond to a signal $\mathcal{R}X$ with output $\mathcal{R}Y$.

**Start:**
- $X$
- $Y$

**Scaled (in time steps):**
- $\mathcal{R}X$
- $\mathcal{R}Y$
**LTI: Linear Time Invariant Systems**

- **Linear:** If a system $H$ responds to a signal $X$ with output $Y$, then system $H$ will respond to a signal $2X$ with output $2Y$.
- **Time-Invariant:** If a system $H$ responds to a signal $X$ with output $Y$, then system $H$ will respond to a signal $\mathcal{R}X$ with output $\mathcal{R}Y$.
- **Take-Away:** If you know how a system responds to a signal a certain time step, you can easily determine how it would respond to a different signal that is a scaled and shifted version of initial signal.

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**System Functionals**

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**System Functional**

- Express a system as a ratio of its output signal over its input signal using operator notation.
- Can make manipulation and synthesis of systems very convenient.

$$Y = \mathcal{R}2X + \mathcal{R}^3X$$

$$H_1 = \frac{Y}{X} = \mathcal{R}2 + \mathcal{R}^3$$
System Functional

- Express a system as a ratio of its output signal over its input signal using operator notation
- Can make manipulation and synthesis of systems very convenient

\[ Y = RX + XY \]

\[ H_1 = \frac{Y}{X} = \frac{R}{1 - R} \]

Combining System Functionals

**System 1:**
\[ X_1 \xrightarrow{H_1} Y_1 \]
\[ H_1 = \frac{Y_1}{X_1} = \frac{R}{1 - R} \]

**System 2:**
\[ X_2 \xrightarrow{H_2} Y_2 \]
\[ H_2 = \frac{Y_2}{X_2} = 5 + R^2 \]

**System Total:**
\[ X_1 \xrightarrow{H_1 \cdot H_2} Y_2 \]

Combining System Functionals

**System Total:**
\[ X_1 \xrightarrow{H_1 \cdot H_2} Y_2 \]

- Recognize what signals are now common: \( Y_1 = X_2 \) for this example
- Multiply Series SFs together (common signals cancel)
- Either convert back to difference equation or keep in this form then

\[ H_{\text{Total}} = \frac{Y_2}{X_1} = H_1 \cdot H_2 = \frac{Y_1}{X_1} \cdot \frac{Y_2}{X_2} = \frac{R}{1 - R} \left( 5 + R^2 \right) = \frac{5R + R^3}{1 - R} \]
Common Topologies

- A few common topologies for linking system functionals can be generalized and then used as needed

- When two systems are in series:

\[ X \xrightarrow{H_1} H_2 \xrightarrow{} Y \]

- Combined form is just multiplication (previous page): 

\[ H_{\text{tot}} = H_1 H_2 \]

---

Common Topologies: Positive Feedback

\[ Y = H_3 (H_2 Y + H_1 X) \]
\[ Y(1 - H_2 Y) = H_1 X \]

\[ X = \frac{H_1}{1 - H_2} \]

\[ \begin{align*}
\frac{Y}{X} &= \frac{n_1 d_2}{d_2 d_1 - n_2 n_1} \\
\frac{Y}{Y} &= \frac{n_1 d_2}{d_2 d_1 - n_2 n_1}
\end{align*} \]

---

Common Topologies: Negative Feedback

\[ Y = H_3 (H_2 Y - H_1 X) \]
\[ Y(1 + H_2 Y) = H_1 X \]

\[ X = \frac{H_1}{1 + H_2} \]

\[ \begin{align*}
\frac{Y}{X} &= \frac{n_1 d_2}{d_2 d_1 + n_2 n_1} \\
\frac{Y}{Y} &= \frac{n_1 d_2}{d_2 d_1 + n_2 n_1}
\end{align*} \]
Common Topologies: Feedforward

\[ Y = H_1X + H_2X \]
\[ Y = (H_1 + H_2)X \]
\[ Y \] = \[ H_1 + H_2 \]
\[ Y \] = \[ \frac{\eta_1 \eta_2 + \eta_2 \eta_2}{\eta_2^2} \]

This formula has no special name

Signals

Unit Sample Signal

* A nudge

\[ \delta[n] = \begin{cases} 
1 & \text{if } n = 0 \\
0 & \text{otherwise} 
\end{cases} \]

Signal is represented with capital Delta: \( \Delta \)
Unit Step Signal

- Turning "on" at n=0

\[ u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \]

Signal is represented with capital U:

\[ U \]

Signals

- Any signal can be made of a sum of appropriately scaled and time-shifted unit sample signals

\[ a[n] = \begin{cases} 1 & \text{if } n=0 \\ 3 & \text{if } n=1 \\ 5 & \text{if } n=2 \\ 0 & \text{otherwise} \end{cases} \]

- More commonly written as:

\[ a[n] = \delta[n] + 3\delta[n-1] - 2\delta[n-2] \]

- And they can be expressed in operator notation: \[ A = \Delta + 3R\Delta - 2R^2\Delta \]

Multiplying Signals and Systems to Get Outputs!!

- If we want to know the output of a system to a certain signal, if they are both in operator notation we only need to multiply

\[ A = \Delta + 3R\Delta - 2R^2\Delta \]

\[ X_1 \rightarrow H_1 \rightarrow Y_2 \]

\[ Y_2 = \frac{V_2}{X_2} = 5 + 3R^2 \]

\[ \text{Output: } A H_2 = (\Delta + 3R\Delta - 2R^2\Delta)(5 + 3R^2) \]

\[ A H_2 = 5\Delta + 15R\Delta - 9R^2\Delta + 3R^2\Delta - 2R^4\Delta \]
Sure Enough...if we simulate it:

\[ H_2 = \frac{V_2}{V_x} = 5 + R^2 \]

\[ AH_2 = 5\Delta + 15R\Delta - 9R^2\Delta + 3R^3\Delta - 2R^4\Delta \]

Superposition

- In previous section we calculated output by distributing out the following: \( AH_2 = (\Delta + 3R\Delta - 2R^2\Delta)(6 + R^2) \)
- This is the same as calculating the output of the system \( H_2 \) to three different inputs separately:
  - \( \Delta \rightarrow \text{Output} = 5\Delta + R^4\Delta \)
  - \( 3R\Delta \rightarrow \text{Output} = 15R\Delta + 3R^3\Delta \)
  - \( -2R^2\Delta \rightarrow \text{Output} = -16R^2\Delta - 2R^4\Delta \)
- ...and then summing them together. Doing it this way is called superposition and is a nice property of LTI systems

\[ AH_2 = 5\Delta + 15R\Delta - 9R^2\Delta + 3R^3\Delta - 2R^4\Delta \]

Superposition

- Graphically can add outputs from individual inputs to get output resulting from complicated input

\[ \Delta \rightarrow \text{Output} = 5\Delta + R^4\Delta \]

\[ 3R\Delta \rightarrow \text{Output} = 15R\Delta + 3R^3\Delta \]

\[ -2R^2\Delta \rightarrow \text{Output} = -16R^2\Delta - 2R^4\Delta \]

\[ AH_2 = 5\Delta + 15R\Delta - 9R^2\Delta + 3R^3\Delta - 2R^4\Delta \]
Feedback

When there is feedback you will get $\mathcal{R}$s in the denominator of your System Functional! These have grave implications. How do we handle that?

- Let's consider the following simplest form of feedback:

\[
\begin{align*}
\gamma[n] &= x[n] + Cy[n - 1] \\
Y &= X + \mathcal{R}CY \\
y^* &= \frac{1}{1 - \mathcal{R}C}
\end{align*}
\]

Let's simulate for different values of $C$.
\[ C = 0.8 \]

\[ \frac{Y}{X} = \frac{1}{1 - RC} \]

\[ C = 0.2 \]

\[ \frac{Y}{X} = \frac{1}{1 - RC} \]

\[ C = 1.2 \]

\[ \frac{Y}{X} = \frac{1}{1 - RC} \]
\[ C = 1.5 \]

\[ Y = \frac{1}{1 - RC} \]

What's going on?

\[ Y = X + CY \]
\[ Y = X + CRX + C^2RY \]
\[ Y = X + CRX + C^2RX^2Y \]

\[ Y \rightarrow Y \]
\[ Y \rightarrow X \]
\[ Y \rightarrow X \]

Convert to Difference Equation:

\[ y[n] = x[n] + Cx[n - 1] + C^2x[n - 2] + \ldots \]
\[ y[n] = \sum_{m=0}^{\infty} C^m x[n - m] \]

• Feedback results in infinite response from an input (signal cycles on the inside)

Plugging in Values

\[ y[n] = \sum_{m=0}^{\infty} C^m x[n - m] \]

• Consider the case where \( x[n] = \delta[n] \):
  • \( y[0] = C^0 \delta[n - 0] + C^1 \delta[n - 1] + C^2 \delta[n - 2] + \ldots \)
  • If \( n < 0 \), \( y[n] = 0 \) (since there's no early non-zero values for the input)
  • \( y[0] = C^0 \delta[0] + C^1 \delta[0] + \ldots = 1 \)
  • \( y[1] = C^1 \delta[1 - 1] = C^1 \)
  • \( y[2] = C^2 \delta[2 - 1] = C^2 \)
  • \( y[3] = C^3 \delta[3 - 1] = C^3 \)
  • \( \ldots \)
  • \( y[0] = C^0 \)

On every time step there is only one non-zero term from the infinite series.
Implications of Terms in the Response

- For the system being looked at: \( y[n] = C^n \)
- In response to a unit sample signal, as \( n \to \infty \):
  - \( Y \) will converge (go towards 0) if \( |C| < 1 \)
  - \( Y \) will diverge (go towards \(+\infty\) or \(-\infty\)) if \( |C| > 1 \)
  - \( Y \) will neither (go towards \(+\infty\) or \(-\infty\)) if \( |C| = 1 \)
- Also:
  - \( Y \) will oscillate if \( C < 0 \) (toggle back and forth)
  - \( Y \) will geometrically grow if \( C > 0 \)
Generalizing this Information

- \( R \) means \( z^{-1} \)
- Find the roots in terms of \( z \) in the denominator. These roots are known as the poles of the system.

\[
y[n] = C y[n-1] + x[n]
\]

\[
y[n] = C^n
\]

\[
x = \frac{1}{1 - RC}
\]

\[
y = x + RCy
\]

\[
y = \frac{1}{1 - RC}
\]

The pole

- \( R \) means \( z^{-1} \)
- Find the roots in terms of \( z \) in the denominator. These roots are known as the poles of the system.

\[
Y(z) = X(z) \frac{1}{1 - z^{-1}C}
\]

\[
Y(z) = X(z) \frac{1}{1 - \frac{z}{z - C}}
\]

\[
Y(z) = X(z) \frac{z}{z - C}
\]

pole at: \( z = C \)

What is this \( z \)??

- From the \( Z \) transform, a discrete time analog of the \( s \) from the Laplace Transform for those of you who are interested...
- Foundation of digital controls, digital signal processing, etc...

Lofti Zadeh... died last week 9/6/2017...

Lofti Zadeh (1921-2017)

*Don’t need to know... just for general interest*
Higher-Order LTI Systems

- A system functional of the form \( \frac{Y}{X} = \frac{A}{1 - RC} \) will have a pole of \( C \).
- Higher order systems can be expanded using partial fraction.
- And it can be revealed that their behavior will be based off of each of the poles that comes from the expanded terms.

General Response

- The response of a system will be based off of the scaled sum of the poles raised to the nth power (the time step):
  \[
  \frac{Y}{X} = \frac{c_1}{1 - RC_1} + \frac{c_2}{1 - RC_2} + \ldots
  \]
  \[y[n] = c_1p^1 + c_2p^2 + \ldots\]
- The \( c \) constants usually come from the residues/top parts of the partial fraction decomposition.

Implications of Poles Response

- Exact response requires knowing all poles AND constants
- Constants are usually residues of partial fraction expansion (annoying to find)
- Thankfully with poles what we're concerned with is usually a different set of questions that don't really need those constants:
- If we know the poles we can:
  - Determine if a system is inherently stable or unstable
  - Determine the nature of the system’s response to general inputs:
    - Is response monotonic?
    - Is response oscillatory?
Don’t generally need PFD
• Poles are easier to find on their own
• They will be the roots of the denominator in z

\[ \frac{Y(z)}{X(z)} = \frac{3z}{1 + 0.6z^{-1} - 0.16z^{-2}} \]

Y(z) = \frac{3z}{1 + 0.6z^{-1} - 0.16z^{-2}} z^2
X(z) = \frac{3z}{1 + 0.6z^{-1} - 0.16z^{-2}}

Poles at: p_1 = -0.8 and p_2 = 0.2

Meaning of Poles
• Poles provide a lot of information!
• The number of poles in a system is what the “order” is.
  • One pole: First-order system
  • Two poles: Second-order system
  • Three poles: Third-order
  • Etc...

Real Poles

\[ y[n] = c_1 p^n + c_2 p^n + c_3 p^n + \ldots \]

• If a pole is real then its unit sample response will be a geometric series. Its sign will dictate response type
  • If \( p > 0 \), the response will oscillate back and forth
  • If \( p < 0 \), the response will oscillate back and forth every other time step
• The magnitude of the pole implies stability:
  • If \( |p| < 1 \) the pole is considered stable and its term’s contribution to a system’s unit sample response will converge to 0
  • If \( |p| > 1 \) the pole is considered unstable and its term’s contribution to a system’s unit sample response will converge to either \( +\infty \) or \(-\infty \) (or both)
  • If \( p = 1 \) its term’s contribution to a system’s unit sample response will converge to its term’s constant value

\[ V_t = Y(z) X(z) \]

\[ Y(z) = \frac{3z}{1 + 0.6z^{-1} - 0.16z^{-2}} \]
Complex Poles

- Poles can also be purely imaginary or complex
- For a system with purely real components, complex poles must occur in complex conjugate pairs
  - Example: $p_1 = 0.5 + j0.3$ and $p_2 = 0.5 - j0.3$
  - Never have a first-order system with a complex pole

- Make more sense in polar form
- Start with $p = a + jb$
- Rewrite as $p = Me^{j\theta}$ where:
  - $M = \sqrt{a^2 + b^2}$
  - $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
- Example $p_1 = 0.5 + j0.3$ and $p_2 = 0.5 - j0.3$
  - $p_1 = 0.583e^{j0.34}$
  - $p_2 = 0.583e^{-j0.34}$

- Since system response is $y[n] = c_1p_1^n + c_2p_2^n + c_3p_3^n + \cdots$
- For complex conj. pair: $y[n] = c(Me^{j\theta})^n + c(Me^{-j\theta})^n$
  - Or $y[n] = c\left(M^n(e^{jn\theta} + e^{-jn\theta})\right)$
  - Or $y[n] = 2cM^n \cos(n\theta)$
  - Or $y[n] \propto M^n \cos(n\theta)$
  - So $Y$ will be oscillatory:
    - Period of oscillation will be $P = \frac{2\pi}{\theta}$
    - Rate of decay of envelope (amplitude of oscillation): $M^n$
  - $M$ is magnitude of pole pair so stab it dictates stability
Poles Together

\[ y[n] = c_0 p^n + c_1 p^n + c_2 p^n + \cdots \]

- In a system, the pole with the largest magnitude is called the "dominant pole". Its behavior dictates the system's "long-term" behavior (as \( n \rightarrow \infty \))
  - If magnitude of dominant pole(s) > 1, system is \textit{unstable} and its response to most bounded inputs will generally diverge/explode
  - If magnitude of dominant pole(s) < 1, system is \textit{stable} and its response to most bounded inputs will converge to a steady value
  - If magnitude of dominant pole(s) = 1, system is \textit{neither stable nor unstable} and its response will be based on its input. Repeated poles at 1 get more complicated
- System will also inherit the behavior of dominant pole(s):
  - If dominant poles are oscillatory, system will tend to oscillate in long-term
  - If dominant pole(s) is real, system will have geometric behavior long-term

Poles Summary in Complex Plane

\[ \text{Re} \quad \text{Imag} \]

- Inside the unit circle: Stable
- Outside the unit circle: Unstable
- Real/positive: Monotonic Response
- Real/negative: Oscillatory (period 2)
- Complex: Oscillatory (period varies)

The Meaning of Poles

- Poles express the way a system wants to behave if left to its own devices
- They are the characteristic values of a system
- They are the \textit{eigenvalues} for those of you with some linear algebra background
- Control/Systems engineering is about manipulating how a system wants to behave (respond faster, with less oscillation, etc...)

There's also zeros... but we won't get into them in 6.01
History
• Steam Engine Governors were first attempt at feedback and control analysis
• All of this stuff we’re covering was developed from the 1920s to the 1960s
• A lot of the math was developed to make battleship guns (WW1) or anti-aircraft fire-systems (WW2) work properly

Application

An Example
• We’re now going to build a discrete time model of a mass sitting on a flat plane:
  • The timestep of the model is $T = 0.1$ seconds
  • The mass of the object is $m = 10$ kg
  • Mass has an acceleration $A$ velocity $V$ a position $D$, and we can apply a force to it $F$
Integrator #1
- Link input force $F$ to velocity $V$

\[
\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

\[
V = \frac{F}{m} \quad \frac{V}{F} = \frac{T}{1 - \lambda}
\]

---

Integrator #2
- Link velocity $V$ to position $D$

\[
\frac{D}{F} = \frac{X_T \frac{F}{m}}{1 - \lambda \frac{F}{m} (1 - \lambda)^2}
\]
Convert Back to Difference Eq if you’d like

\[
\frac{D}{P} = \frac{x^2T^2}{m(1-\frac{x}{m})^2}
\]

\[
d[n] - 2d[n-1] + d[n-2] = \frac{T^2}{m} f[n-2]
\]

\[
d[n] = d[n-1] + d[n-2] + \frac{T^2}{m} f[n-2]
\]

Poles of Double Integrator

• A discrete time integrator has one pole at 1. Two in series will have two poles at 1

\[
D(z) = \frac{T^2}{m(z-1)(z-1)}
\]

\[
p_1 = 1 \quad p_2 = 1
\]

Nudge it with a Unit Sample

\[
T = 0.1 \text{ and } m = 10
\]

\[
Poles:
\]

\[
p_1 = 1 \quad p_2 = 1
\]
Double Delayed Nudge

\[ T = 0.1 \text{ and } m = 10 \]

Poles:
\[ p_1 = 1 \quad p_2 = 1 \]

Superposition provides visual support for why the response is what it is.

Adding in Some Friction

- We're going to model friction now
- Opposing force proportional to velocity
Adding in Some Friction

\[ V = \frac{F_e}{m} + \frac{R T}{m} V \]

\[ V = \frac{R T}{1 - R \left(1 - \frac{T}{m}\right)} \]

\[ D = \frac{R T}{1 - R} \]

\[ P = \frac{R T}{1 - R} \frac{T^2}{m} \]

Adding in Some Friction

\[ V = \frac{F_e}{m} + \frac{R T}{m} V \]

\[ D = \frac{R T}{1 - R} \]

\[ P = \frac{R T}{1 - R} \frac{T^2}{m} \]

Poles with Friction

- The friction is a “lossy” mechanism, meaning the pole of the force-to-velocity system is now less than 1.
- The pole from the velocity-to-position system is still representative of a perfect discrete time integrator.
Convert Back to Difference Eq (if you want)

\[ P = \frac{\frac{S^2 T^2}{m}}{(1 - R (1 - K m^{-1})) (1 - R)} \]

\[ d(n) = \left(2 - K \frac{T}{m} \right) d[n - 1] + \left(1 - K \frac{T}{m} \right) d[n - 2] + \frac{T^2}{m} \delta[n - 1] \]

\[ d(n) = \left(2 - K \frac{T}{m} \right) d[n - 1] - \left(1 - K \frac{T}{m} \right) d[n - 2] + \frac{T^2}{m} \delta[n - 1] \]

Nudge it with a Unit Sample

\[ T = 0.1 \text{ and } m = 10 \]
\[ K_P = 1.0 \]

Poles:
\[ p_2 = 1 \quad p_1 = \left(1 - K \frac{T}{m} \right) \]

Nudge it with Double Unit Sample

\[ T = 0.1 \text{ and } m = 10 \]
\[ K_P = 1.0 \]

Poles:
\[ p_2 = 1 \quad p_1 = \left(1 - K \frac{T}{m} \right) \]
Adding in Spring Return

- Force that tends to generate return force proportional to position
- Positive Distance leads to negative spring force

\[
\begin{align*}
H_1 &= \frac{\rho^2 T^2}{(1 - R(1 - \frac{\rho}{\rho_0}))(1 - R)}
\end{align*}
\]
\[
\begin{align*}
\frac{D}{F_0} = \frac{H_1}{1 + K_s H_1} & \quad \text{H}_1 = \frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R} (1 - \frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}})} \\
\text{Remembering in feedback that:} & \quad \frac{F}{X} = \frac{n_2 d_2}{d_2 n_2 + n_2 n_1} \\
\frac{D}{F_0} = \frac{H_1}{1 + K_s H_1} & \quad = \frac{n_2 d_2}{d_2 n_2 + n_2 n_1} = \frac{\left(\mathcal{R}^2 T^2_m\right) (1)}{\left(1 - \mathcal{R} \left(1 - \frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right)\right) + K_s \left(\mathcal{R}^2 T^2_m\right)} \\
\frac{D}{F_0} = \frac{H_1}{1 + \mathcal{R} \left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right) + K_s \left(\mathcal{R}^2 T^2_m\right)}
\end{align*}
\]

\[
\begin{align*}
\frac{D(z)}{F_0(z)} & = \frac{z - 2 \mathcal{R}^2 T^2_m}{1 + z^{-1} \left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right) + z^{-2} \left(1 - \mathcal{R} \left(1 - \frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right)\right) + K_s \left(\mathcal{R}^2 T^2_m\right)} \\
\frac{D(z)}{F_0(z)} & = \frac{z - 2 \mathcal{R}^2 T^2_m}{1 + z^{-1} \left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right) + z^{-2} \left(1 - \mathcal{R} \left(1 - \frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right)\right) + K_s \left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right)} \\
\frac{D(z)}{F_0(z)} & = \frac{z - 2 \mathcal{R}^2 T^2_m}{z^2 + z \left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right) + \left(1 - \mathcal{R} \left(1 - \frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right)\right) + K_s \left(\mathcal{R}^2 T^2_m\right)} \\
\frac{D(z)}{F_0(z)} & = \frac{- \left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}} - 2\right) \pm \sqrt{\left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}} - 2\right)^2 - 4 \left(1 - \mathcal{R} \left(1 - \frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right)\right)}}{2}
\end{align*}
\]

Special Cases

\[
\begin{align*}
p_{1,2} & = \frac{- \left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}} - 2\right) \pm \sqrt{\left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}} - 2\right)^2 - 4 \left(1 - \mathcal{R} \left(1 - \frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}\right)\right)}}{2} \\
\text{If } K_s = 0, \text{...simplifies to:} & \\
p_{1,2} & = \frac{- \left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}} - 2\right) \pm \sqrt{\left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}} - 2\right)^2}}{2} \\
\text{and} & \\
p_{1,2} & = \frac{- \left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}} - 2\right) \pm \sqrt{\left(\frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}} - 2\right)^2}}{2} \\
p & = \frac{1}{p_2} = 1 - \frac{\mathcal{R}^2 T^2_m}{1 - \mathcal{R}}
\end{align*}
\]
Special Cases

\[ p_{1,2} = \frac{-(K_f T_m - 2) \pm \sqrt{(K_f T_m - 2)^2 - 4\left(1 - K_f T_m + K_s T_m^2\right)}}{2} \]

- If \( K_f = 0.0, K_s = 0 \ldots \) simplifies to:

\[ p_{1,2} = \frac{-(2) \pm \sqrt{-4K_s T_m^2}}{2} \]

\[ z_{1,2} = 1 \pm j\sqrt{\frac{4K_s}{T_m}} \]

Convert back to Difference Eq if you’d like!

\[ p = \frac{T_m^2}{1 + j\left(K_f T_m - 2\right) + j\left(1 - K_f T_m + K_s T_m^2\right)} \]

\[ p\left(1 + j\left(K_f T_m - 2\right) + j\left(1 - K_f T_m + K_s T_m^2\right)\right) = F_0 \left(T_m^2\right) \]

\[ p[n] = \left(2 - K_f T_m\right)p[n-1] - \left(1 - K_f T_m + K_s T_m^2\right)p[n-2] + \frac{T_m^2}{m} p[n-2] \]

Specific Case

\[ p_{1,2} = \frac{-(K_f - 3) \pm \sqrt{(K_f - 3)^2 - 4\left(1 - K_f T_m + K_s T_m^2\right)}}{2} \]

- \( K_f = 1.0, K_s = 1.0, T = 0.1, \) and \( m = 10 \ldots \) input unit sample

- Theory predicts:
  - \( p_{1,2} = 0.95 \pm \sqrt{0.0866j} \) or \( p_1 = 0.9539e^{\sqrt{0.0866j}} \) and \( p_2 = 0.9539e^{-\sqrt{0.0866j}} \)
  - Meaning:
    - Oscillatory response
    - Decays to zero (magnitude less than 1)
    - Period of oscillation ~ 69.114 time steps

Prediction Correct
Specific Case

- $k_f = 0.1$, $k_s = 1.0$, $T = 0.1$, and $m = 10$...input unit sample

- Theory predicts:
  - $p_1, p_2 = 0.99 \pm 0.009$ or $p_1 = 0.999e^{-0.110}$ and $p_2 = 0.995e^{-0.110}$
  - Meaning:
    - Oscillatory response
    - Bigger pole magnitude so decay takes longer
    - Decays to zero (magnitude less than 1)
    - Period of oscillation ~ 62.7 time steps

Where is this overall behavior coming from?

- $y[n] = c_1 \times 0.999^n + c_2 \times 0.911^n$
- $c_2$ must be negative.
Conclusions

- System Functionals provide a convenient and cleaner way of combining systems with systems and systems with signals.
- Poles provide a way to characterize a system response without necessarily having to simulate it!
- Designing with poles in mind allows us to “quickly” get the response we want.

Trends

\[
p_{1,2} = -\left(\frac{K_f}{T_m} - 2\right) \pm \sqrt{\left(\frac{K_f}{T_m} - 2\right)^2 - 4\left(1 - \frac{K_f}{T_m} + \frac{K_s}{T_m}\right)}
\]

- What relationship between \(K_f\) to \(K_s\) is needed to guarantee no oscillation in responses? (assume \(T = 0.1\), and \(m = 10\))

Expand under root

\[
p_{1,2} = -\left(\frac{K_f}{T_m} - 2\right) \pm \left(\frac{\left(\frac{K_f}{T_m} - 2\right)^2 - 4\left(1 - \frac{K_f}{T_m} + \frac{K_s}{T_m}\right)}{2}\right)
\]

Expand under root

\[
\left(\frac{K_f}{T_m}\right)^2 > K_s > \frac{K_f^2}{T_m} \Rightarrow \frac{K_f^2}{T_m} > K_s \Rightarrow K_s > mK_s
\]

Finally

- Can modify this to affect entire system!
- Universe gives us this

\[
V_c \xrightarrow{H_{Controller}} V_s \xrightarrow{H_{Plant}} V_o
\]